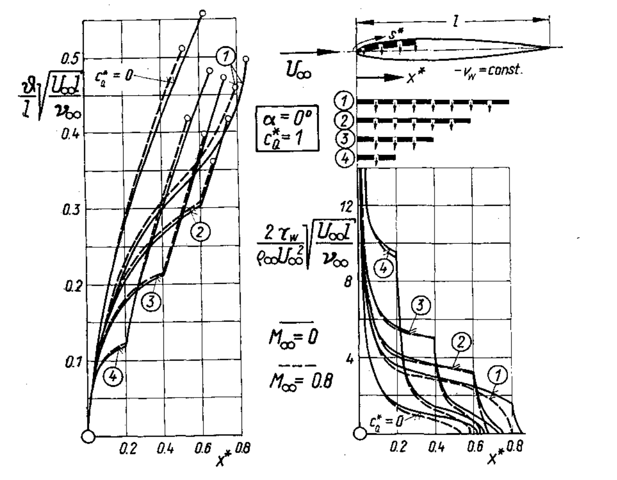

Fig. 1 Laminar boundary layer with continuously distributed suction on profile NACA 0010

$R(x)$, and B also of $v_w(x)$. Having calculated the momentum-loss thickness from Eq. (8), one can evaluate the other boundary layer quantities as just explained.

The method has been checked against special exact solutions, namely, the flat plate and the "similar solutions." In the large range of pressure gradients and suction parameters investigated, good agreement with the exact theory was found.

Furthermore, systematic calculations have been performed for airfoils in the subsonic and supersonic ranges. By means of numerical examples, the influence of the intensity and location of suction on the development of the boundary layer was studied. In Fig. 1 some results are presented for the subsonic airfoil NACA 0010. A given suction quantity

$$c_Q^* = \left(\frac{U_\infty l}{\nu_\infty}\right)^{1/2} \int_0^{s_t^*} \frac{-v_w \rho_w}{U_\infty \rho_\infty}\, ds^* \qquad (10)$$

(where $s^* = s/l$ = coordinate along the contour, and subscript t = trailing edge) is distributed in various ways as indicated in the figure. With decreasing suction zone, the intensity v_w must be increased so that c_Q^* remains constant. For compressible flow, v_w must be increased to compensate for the reduced density ρ_w at the wall. The positions x_{sep}^* of the separation points for the various suction distributions of Fig. 1 are plotted in Fig. 2a. It is demonstrated that by extending the suction zone to the rear part of the profile the point of separation moves in the same direction. For compressible flow, separation occurs somewhat farther upstream. In Figs. 2c and 2d, the same graphs are given for larger angles of incidence. In Fig. 2b, the case is treated in which a suction zone of constant length is situated at various positions along the upper surface. Here, too, suction in the rear part is seen to be more effective in delaying separation than nose suction.

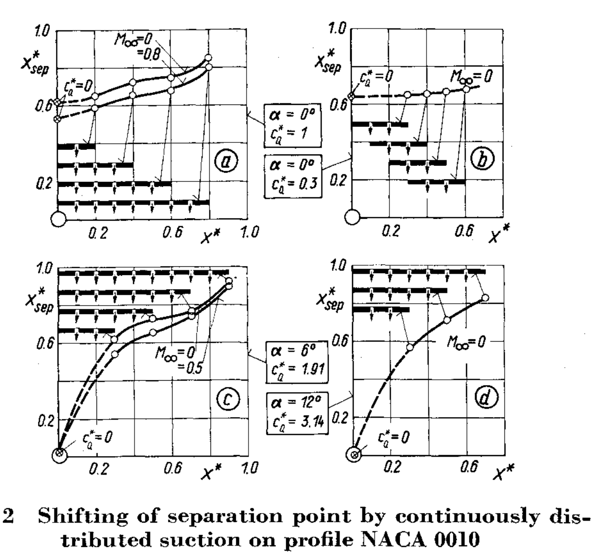

Fig. 2 Shifting of separation point by continuously distributed suction on profile NACA 0010

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Quasi-Steady Aspects of the Adjustment of Separated Flow Regions to Transient External Flows

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The adjustment of separated flow regions with respect to pressure, mass, and heat transfer to transient external flows is investigated using several characteristic times. The adequacy of a quasi-steady treatment is demonstrated, and a quasi-steady solution for the supersonic two-dimensional base pressure problem is presented as an illustration. Experimental data are presented for a transient external flow and compared to the quasi-steady solution for this case. It is concluded that, although initial response to pressure waves is very rapid, the adjustment due to mass and heat transfer is much slower, and, as a result, short-duration experiments on separated flows should come under special scrutiny for correct interpretations of results.

The dependence of wake problems on both the dynamics of the external, nearly isentropic stream and the dissipative mechanism of jet-mixing regions is well established.[1–3]

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In particular, the mixing component has received considerable attention since it has assumed not only a predominate role in the treatment of the separated (wake) flows but, more recently, also in the problem of flow separation itself.[4–6] As quantitative information on the mechanism within the mixing regions has become available,[7,8] the analysis of transport processes controlled by jet mixing could be carried out for both single stream[9] and two-stream cases.[10] Although all such studies were restricted to steady flow jet mixing, the quantitative aspects of rate processes controlled by the mixing component of wake flows suggested that the response of wake flows to transient flow conditions would lead to two distinctively different characteristic response times for adjustment due to pressure waves on one hand and transport phenomena on the other. The characteristic time for wake adjustment due to pressure waves can be expressed by

$$\Delta t_a = L/c_a$$

where L is a characteristic length dimension of the wake and c_a is a reference acoustic velocity. A characteristic adjustment time for mass transfer could be expressed by

$$\Delta t_m = L\sigma/u_a I_{1,j}$$

where σ is the mixing parameter[8] (tentatively, $\sigma = 12 + 2.76M_a$), u_a is the flow velocity in the freestream adjacent to the mixing region, and $I_{1,j}$ is a mass flow integral defined for a mixing region by Eq. (11).

The ratio of the characteristic times is then

$$\Delta t_m/\Delta t_a = \sigma/M_a I_{1,j}$$

a function of the Mach number along the mixing region only. Now for $M_a \approx 1$, this ratio will be nearly 30, and this suggests that a quasi-steady concept of wake response due to transient external stream conditions can be justified under such, or similar, conditions, inasmuch as pressure equalization across the mixing region can be considered as instantaneous compared to the much slower process of mass exchange toward the equilibrium condition of the wake with conservation of mass in the wake. It will be noted that heat transfer processes across wakes are even more sensitive to the controlling effects of mixing regions, as the characteristic time for heat transfer now is expressed by

$$\Delta t_H = L/u_a St$$

and

$$(\Delta t_H/\Delta t_a) = (1/M_a St)$$

So that with a Stanton number $St \approx 0.01$[9] and $M_a \approx 1$

$$\Delta t_H/\Delta t_a \approx 100$$

This suggests that heat transfer experiments on separated flow regions, although tending to show a rapid response to pressure-induced flow conditions, actually will have a very slow adjustment towards equilibrium temperatures.‡

A quasi-steady analysis of certain transient wake problems then may be justified by the foregoing considerations. As an illustration for the validity of such concepts, the steady-state solution of the supersonic two-dimensional base pressure problem is extended to include cases where the approaching stream has transient properties. The following analysis is for the simple backstep geometry (Fig. 1) and is restricted to cases where the approaching boundary layer is negligible or where the mixing profile is fully developed, so that the "restricted theory"[9] can be used. According to the stipulated quasi-steady aspects of the problem, all time derivatives in dynamic solutions are discarded, whereas time-dependent terms in the continuity and energy equations are retained. The transient properties of the approaching stream are the Mach number M_{1a} (or Crocco number C_{1a}), stagnation pressure P_{01a}, and stagnation temperature T_{01a}. It is assumed also that, at any instant, the stagnation temperature is essentially constant throughout the flow region. (Subscript 1 refers to location 1 just before the expansion of free-stream, whereas the subscript a refers to the isentropic flow adjacent to the dissipative regions.) The continuity equation for nonsteady conditions is

$$\frac{\partial w}{\partial t} = -\int_A \rho u \cdot dA \qquad (1)$$

Equation (1) is applied to the entire wake region in which the mass

$$w = \rho_b V \text{ lbm/unit width} \qquad (2)$$

is considered to have the uniform density of ρ_b (subscript b for bulk), whereas the total entrained volume V (per unit width) can be expressed as

$$V = \tfrac{1}{2}H^2 \cot\theta_{2a} + B \qquad (3)$$

Here θ_{2a} is the deflection angle of the free jet after it has expanded around the edge of the backstep, H is the step height, and B is the volume per unit width of any part of the entrained wake other than the triangular shaped region. The density of the bulk of the mass of the wake can be related to the freestream stagnation conditions and the freestream Mach number (here expressed in terms of Crocco number). Equations (2) and (3) can be expressed as

$$w = (P_{01a}/RT_{01a})(1 - C_{2a}{}^2)^{k/(k-1)}(\tfrac{1}{2}H^2\cot\theta_{2a} + B) \qquad (4)$$

where subscript 2 refers to flow conditions after the expansion.

Equation (4) is differentiated with respect to time t so as to treat all the freestream and free jet variables as time dependent. If one can assume that the change in θ_{2a} will not affect the total volumes materially, one obtains upon differentiation

$$\frac{dw}{dt} = \frac{V}{RT_{01a}}(1 - C_{2a}{}^2)^{k/(k-1)}\left[\frac{dP_{01a}}{dt} - \frac{P_{01a}}{T_{01a}}\frac{dT_{01a}}{dt} - \frac{P_{01a}}{(1 - C_{2a}{}^2)}\left(\frac{k}{k-1}\right)\frac{dC_{2a}{}^2}{dt}\right] \qquad (5)$$

giving an expression for the rate of change of mass in the wake region. This, in turn, must be equal to the flow out the wake through the recompression region located at the end of the wake between cross sections 3 and 4

$$\frac{dw}{dt} = G_d = \int_{y_j}^{y_d} \rho u dy \qquad (6)$$

This integral describes the flow for unit width between the jet boundary stream line, subscript j, originating at the separa-

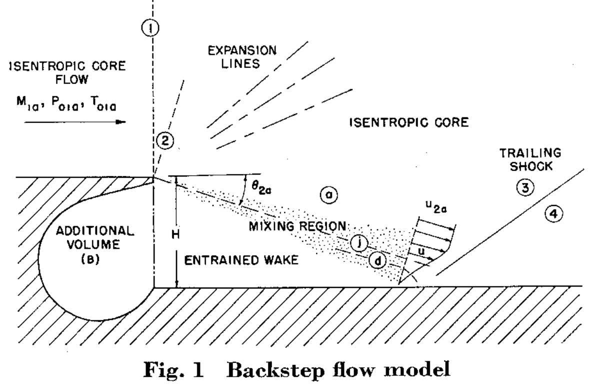

Fig. 1 Backstep flow model

‡ The reliance of shock tube experiments for studying heat transfer rates in separated flow regions without checking the response of bulk temperatures in a wake over longer periods of time (for correct interpretation of short duration experiments) can be criticized strongly on this account.

tion point and the discriminating stream line, subscript d, which satisfies the escape criterion[9]

$$P_{03d}/P_3 = P_4/P_3 = (1 - C_{3d}^2)^{-k/(k-1)} \quad (7)$$

where

$$P_4/P_3 = (P_4/P_3)(M_{2a}, \theta_{4a} - \theta_{2a}, k) \quad (8)$$

is determined by the oblique shock relations applied to the adjacent freestream at the end of the wake. With the help of Eqs. (7) and (8), C_{3d} may be calculated for a given expansion angle θ_{2a}.

The "restricted theory"[9] uses the error function, velocity distribution for describing the flow in the jet mixing region, so that in dimensionless representation

$$\varphi = u/u_{2a} \qquad \eta = \sigma(y/x)$$
$$\varphi(\eta) = \tfrac{1}{2}(1 + \mathrm{erf}\,\eta) \quad (9)$$

For isoenergetic flow $T_0(\eta) = T_{01a}$ and for an isobaric mixing process, the mass flow between the stream lines j and d can be expressed by

$$G_d = \rho_{2a}u_{2a}(x/\sigma)(1 - C_{2a}^2)(I_{1,d} - I_{1,j}) \quad (10)$$

where

$$I_1 = I_1(C_{2a}^2,\eta) = \int_{-\infty}^{\eta} \frac{\varphi}{1 - C_{2a}^2\varphi^2}\,d\eta \quad (11)$$

is available in tabulated form.[7] With C_{3d} determined by Eqs. (7) and (8), one finds $\varphi_{3d} = C_{3d}/C_{2a}$, where $C_{2a} = C_{3a}$ for $P_2 = P_3$ and obtains η_d with Eqs. (9), whereas $I_{1,j}$ is a function only of C_{2a} and is tabulated in Ref. 7. For the isentropic adjacent freestream

$$\rho_{2a}u_{2a} = \frac{P_{01a}}{(T_{01a})^{1/2}}\left[\frac{R}{kg}\,\frac{2}{k-1}\,\frac{C_{2a}^2}{(1 - C_{2a}^2)^2} \times (1 - C_{2a}^2)^{2k/(k-1)}\right]^{1/2} \quad (12)$$

and for the simple backstep

$$x = H/\sin\theta_{2a} \quad (13)$$

Substitution of Eqs. (12) and (13) into Eq. (10) yields

$$G_d = \frac{HC_{2a}}{\sigma\,\sin\theta_{2a}}\left(\frac{2}{k-1}\right)^{1/2}(1 - C_{2a}^2)^{k/(k-1)}\,P_{01a}\left(\frac{kg}{RT_{01a}}\right)^{1/2}(I_{1,d} - I_{1,j}) \quad (14)$$

Combining Eqs. (5) and (14), one obtains finally

$$\frac{d}{dt}C_{2a}^2 = \frac{k-1}{k}(1 - C_{2a}^2)\left[\frac{1}{P_{01a}}\frac{dP_{01a}}{dt} - \frac{1}{T_{01a}}\frac{dT_{01a}}{dt} - \frac{C_{2a}}{\sigma\,\sin\theta_{2a}}\,\frac{H}{V}\left(\frac{2kgRT_{01a}}{k-1}\right)^{1/2}(I_{1,d} - I_{1,j})\right] \quad (15)$$

Integration of differential equation (15) for given "forcing functions" $P_{01a}(t)$ and $T_{01a}(t)$ with a given initial condition $\theta_{2a}(t = 0)$ taken, e.g., from the steady-state solution, yields $C_{2a}(t)$ and subsequently the base pressure ratio,

$$(P_b/P_{01a})(t) = [1 - C_{2a}^2(t)]^{k/(k-1)}$$

An experimental program was carried out in the supersonic facility of the Departments of Mechanical and Aeronautical Engineering at the University of Illinois to check the validity of the quasi-steady concepts that form the basis of the foregoing analysis. The test configuration provided two-dimensional flow and an approaching freestream Mach number of $M_{1a} = 1.95$ ($C_{1a} = 0.657$) and a test section providing a configuration as shown in Fig. 1, where $H = 0.55$ in. and the two-dimensional channel was 1 in. deep.

While other flow parameters were held constant, the stagnation pressure P_{01a} was lowered to about 60% of its initial constant value during periods of time ranging from approximately 0.06 to 2.5 sec by means of a pneumatically operated pressure reducing valve. Time histories of base pressures P_b and stagnation pressures P_{01a} were recorded by means of transducers and a recording oscillograph. The agreements between theory and experiment appear to be satisfactory, particularly when the volume of the entrained wake was close to that of a simple backstep. This is shown in Fig. 2, which is representative for a large number of tests carried out under varying transient conditions in the freestream. The agreement between experimental data and analytical results based on a quasi-steady theory applied to the transient base pressure problem suggests the following conclusions:

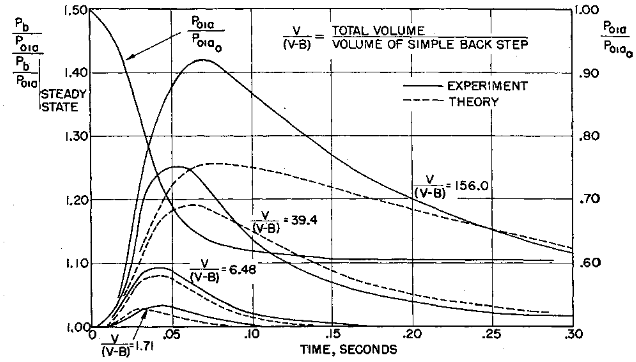

Fig. 2 Transient influence on specific base pressure ratio

1) The jet mixing component effectively controls the rate of exchange processes across separated flow regions.

2) The wake adjustment due to mass transfer, and due even more to heat transfer, is much slower than the rapid initial response to pressure waves.

3) Short duration experiments conducted for studying separated flows with pressure response and transport phenomena being subjected to entirely different time scales toward steady flow establishment must be met with special scrutiny for obtaining correct interpretations of results. This, indeed, has been borne out by experiments.[11]

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Free-Convection Laminar Boundary Layers in Oscillatory Flow

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THE study of laminar boundary layers in oscillatory flow with a steady mean was initiated by Lighthill,[1] who considered the effect of fluctuations of freestream velocity on the skin friction and heat transfer for plates and cylinders. Since then various aspects of this problem have been considered by many workers.[2–4] The present note considers the corresponding free-convection problem for a vertical flat plate, when the plate temperature oscillates in time about a constant nonzero mean while the freestream is isothermal. The boundary layer equations, in terms of stream function by which the continuity equation is identically satisfied, are, in dimensionless form,

$$\psi_{yt} + \psi_y\psi_{xy} - \psi_x\psi_{yy} = G + \psi_{yyy} \tag{1}$$

$$G_t + \psi_y G_x - \psi_x G_y = (1/\sigma)G_{yy} \tag{2}$$

where σ is the Prandtl number. The boundary conditions to be satisfied are

$$\begin{array}{lll} y = 0 & \psi_y = \psi_y = 0 & \\ G = G_0(1 + \epsilon e^{i\omega t}) & & \epsilon \ll 1 \\ y \to \infty & \psi_y \to 0 & G \to 0 \end{array} \tag{3}$$

Now write ψ and G as the sum of steady and small oscillating components:

$$\begin{aligned} \psi &= A(x,y) + \epsilon e^{i\omega t} B(x,y) \\ G &= P(x,y) + \epsilon e^{i\omega t} Q(x,y) \end{aligned} \tag{4}$$

where (A,P) is the steady mean flow and satisfies

$$\begin{array}{lll} A_yA_{xy} - A_xA_{yy} = P + A_{yyy} & & \\ A_yP_x - A_xP_y = (1/\sigma)P_{yy} & & \\ y = 0 & A_x = A_y = 0 & P = G_0 \\ y \to \infty & A_y \to 0 & P \to 0 \end{array} \tag{5}$$

Neglecting squares of ϵ and dividing by $e^{i\omega t}$, one finds that (B,Q) satisfy the following differential set:

$$\begin{array}{lll} i\omega B_y + B_yA_{xy} + A_yB_{xy} - A_xB_{yy} - B_xA_{yy} = B_{yyy} + Q & & \\ i\omega Q + A_yQ_x + B_yP_x - A_xQ_y - B_xP_y = (1/\sigma)Q_{yy} & & \\ y = 0 & B_x = B_y = 0 & Q = G_0 \\ y \to \infty & B_y \to 0 & Q \to 0 \end{array} \tag{6}$$

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Method of Solution

Considering set (5), one finds that this is the boundary layer problem of steady free-convection flow over a vertical plate and can be reduced to ordinary differential equations by the similarity transformation

$$\begin{aligned} \eta &= y(G_0/x)^{1/4} \\ A &= 4(G_0x^3)^{1/4}F(\eta) \\ P &= G_0\theta(\eta) \end{aligned} \tag{7}$$

Set (6) is considered next. It is convenient to write B and Q as sums of in-phase and out-of-phase components. Substitute $B = M + iN$, $Q = R + iS$ in Eq. (6) and separate real and imaginary parts to get

$$\begin{array}{lll} -\omega N_y + M_yA_{xy} + A_yM_{xy} - A_xM_{yy} - M_xA_{yy} = M_{yyy} + R & & \\ -\omega S + A_yR_x + M_yP_x - A_xR_y - M_xP_y = (1/\sigma)R_{yy} & & \\ y = 0 & M_x = M_y = 0 & R = G_0 \\ y \to \infty & M_y \to 0 & R \to 0 \end{array}$$

and

$$\begin{array}{lll} \omega M_y + N_yA_{xy} + A_yN_{xy} - A_xN_{yy} - N_xA_{yy} = N_{yyy} + S & & \\ \omega R + A_yS_x + N_yP_x - A_xS_y - P_yN_x = (1/\sigma)S_{yy} & & \\ y = 0 & N_x = N_y = S = 0 & \\ y \to \infty & N_y \to 0 & S \to 0 \end{array} \tag{8}$$

Low-Frequency Oscillations

Similarity solutions of the partial differential set (8), as found in the case of set (5), do not exist. However, for low-frequency oscillations the series expansions

$$M = \left(\frac{x_1}{\omega^2}\right)^{3/4} \sum_{p=1}^{\infty} M_p(\eta)x_1^{p-1}$$

$$N = \left(\frac{x_1^5}{\omega^6}\right)^{1/4} \sum_{p=1}^{\infty} N_px_1^{p-1}$$

$$R = \sum_{p=1}^{\infty} R_p(\eta)x_1^{p-1}$$

$$S = (x_1)^{1/2} \sum_{p=1}^{\infty} S_p(\eta)x_1^{p-1}$$

where $x_1 = (x\omega^2/G_0)$ may be introduced. Substituting in (8) and equating powers of x_1, one obtains the following set of ordinary equations:

$$\begin{aligned} M_1''' + 3FM_1'' - 4F'M_1' + 3F''M_1 &= -R_1 \\ (1/\sigma)R_1'' + 3FR_1' &= -\tfrac{3}{4}\theta'M_1 \\ N_1''' + 3FN_1'' - 6F'N_1' + 5F''N_1 &= M_1' - S_1 \\ (1/\sigma)S_1'' + 3FS_1' - 2F'S_1 &= R_1 - \tfrac{5}{4}\theta'N_1 \end{aligned} \tag{9}$$

$$\begin{aligned} M_n''' + 3FM_n'' - 4nF'M_n' + (4n-1)F''M_n &= -R_n - N_n' \\ (1/\sigma)R_n'' + 3FR_n' - 4(n-1)F'R_n &= -S_{n-1} - (n - \tfrac{1}{4})\theta'M_n \\ N_n''' + 3FN_n'' - (4n+2)F'N_n' + (4n+1)F''N_n &= M_n' - S_n \\ (1/\sigma)S_n'' + 3FS_n' - (4n-2)F'S_n &= R_n - (n + \tfrac{1}{4})\theta'N_n \end{aligned} \tag{10}$$

where

$$n = 2, 3, 4, \ldots$$